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Corner Transfer Matrices of the Eight-Vertex Model. I. Low-Temperature Expansions and Conjectured Properties

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A "corner transfer matrix" (CTM) is defined for the zero-field, eight-vertex model on the square lattice. Its logarithm and its diagonal form are obtained to second order in a perturbation expansion of low-temperature type. They turn out to have a very simple form, apart from certain "remainder" contributions that can be ignored in the limit of a large lattice. It is conjectured that in this limit the operators have these simple forms for all temperatures less than the critical temperature T_c. The spontaneous magnetization can then easily be obtained, and agrees with the expression previously proposed. It is intended to prove some of the conjectures in subsequent papers.

KEY WORDS: Lattice statistics; eight-vertex model; corner transfer matrices; spontaneous magnetization.

1. INTRODUCTION AND SUMMARY

At present there are two models in lattice statistics for which the free energy has been obtained exactly, but the spontaneous magnetizations or polarizations have only been obtained as a conjecture (albeit a rather plausible one). The models are the eight-vertex model on the square lattice[1,2] and the pure three-spin interaction model on the triangular lattice.[3,4]

It is possible that there is a more general model that is exactly solvable and includes both these (together with the normal Ising model on the triangular lattice). However, one has not yet been found, so for the moment we must consider the eight-vertex and three-spin models separately.

One way of attempting to calculate the magnetization and polarization of the eight-vertex model would be to take the known form[5] of the eigenvectors of the row-to-row transfer matrix and use methods similar to those

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previously employed for the six-vertex model.[6] However, even if this rather ambitious program could be carried through, it would not be very illuminating.

The purpose of the present and subsequent papers is to explore another approach, which promises to give some insight into the structure of the operators that build up the partition function, and perhaps provide a completely new way of obtaining the thermodynamic properties of the system. For the moment we shall consider only the eight-vertex model, but similar techniques should be applicable to the three-spin model.

In Sections 2–4 we define two "corner transfer matrices" A and B for the square lattice. (For an isotropic lattice they are the same.) In Section 5 we obtain second-order perturbation expansions of low-temperature type for $\ln A$ and for the diagonal form A_d of A. Apart from certain "remainder" or "boundary" contributions that can be ignored in the thermodynamic limit, they have very simple forms: A_d is a direct product of diagonal two by two matrices [Eq. (50)] and $\ln A$ is an anisotropic Heisenberg chain operator, but with coefficients proportional to the site number j [Eq. (58)]. The same is true for B, and A and B commute.

In Section 6 we introduce the usual elliptic function parameters q, x, and z that occur in the calculation of the free energy of the eight-vertex model.[1] Then in Section 7 we conjecture that Eqs. (50) and (58) are in fact exact for all temperatures less than T_c, and specify the coefficients κ, w, w', $\mathscr{J}_1$, $\mathscr{J}_2$, $\mathscr{J}_3$, and $\mathscr{J}_4$ occurring in the equations. The spontaneous magnetization is then readily obtained [Eq. (57)] and agrees with the previous conjecture of Barber and Baxter.[2]

Finally, in Section 8 we discuss the significance of these conjectures, and remark that some partial results have been obtained to support them. It is intended to give these in subsequent papers.

Notation

The symbols a, b, c, and d are used throughout this paper for the Boltzmann weights of the eight-vertex model; they are scalars and are never given a suffix. The symbols s_j, c_j, and d_j are used for the two by two Pauli spin matrices [Eq. (16)]; they always have a suffix.

2. CORNER TRANSFER MATRICES

Consider the spin formulation of the eight-vertex model.[7,8] Label the columns of the square lattice by i, the rows by j. Let $\sigma_{i,j} = \pm 1$ be the spin associated with the site on column i and row j. Assign diagonal pair inter-

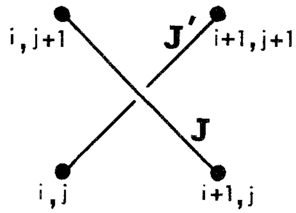

Fig. 1. The diagonal interactions between spins on the lattice. There is also a four-spin interaction.

actions $-J$, $-J'$ as in Fig. 1, together with a four-spin interaction $-J_4$. Then the Hamiltonian is

$$H = -\sum_i \sum_j (J\sigma_{i,j+1}\sigma_{i+1,j} + J'\sigma_{ij}\sigma_{i+1,j+1} + J_4\sigma_{ij}\sigma_{i+1,j}\sigma_{i,j+1}\sigma_{i+1,j+1}) \quad (1)$$

and the partition function is

$$Z = \sum \exp(-\beta H) \quad (2)$$

where the summation is over all allowed values of the spins, and $\beta = 1/k_B T$. We take J and J' to be positive.

Consider the lattice of sites (i, j) such that $|i| + |j| \leqslant n + 2$, excluding $(0, \pm(n + 2))$ and $(\pm(n + 2), 0)$. This lattice has diagonal boundaries, as shown in Fig. 2. Impose the boundary condition

$$\sigma_{ij} = +1 \qquad \text{if} \quad |i| + |j| = n + 1 \text{ or } n + 2 \quad (3)$$

Divide the lattice by two cuts into four quadrants of equal size, as shown in Fig. 2. Perform the summation in (2) over all spins, *except for those on the cuts.*

Let τ_1 denote the n spins on the right half (including the center) of the horizontal cut. Similarly, let τ_2, τ_3, and τ_4 denote the spins on the half-cuts

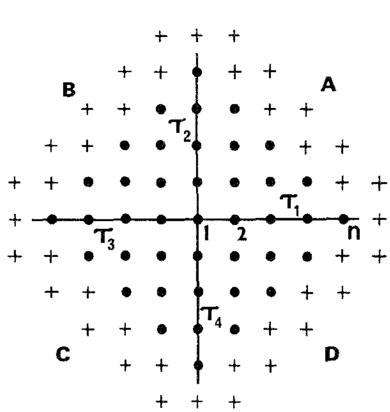

Fig. 2. The lattice with diagonal boundaries, for the case $n = 5$. Spins on sites shown by filled circles are free; those shown by a plus sign are fixed to be up, i.e., to have value $+1$.

(including the center) indicated in Fig. 2. If $\tau_1,..., \tau_4$ are held fixed, then clearly the summation in (2) factors into four parts, and (2) becomes

$$Z = \sum_{\tau_1} \sum_{\tau_2} \sum_{\tau_3} \sum_{\tau_4} A(\tau_1|\tau_2)B(\tau_2|\tau_3)C(\tau_3|\tau_4)D(\tau_4|\tau_1) \tag{4}$$

where $A(\tau_1, \tau_2)$ is the contribution to Z from the upper right quarter of the lattice, B is the contribution from the upper left, etc.

We can write A explicitly. Draw the upper right quarter of the lattice as in Fig. 3. Label the spins on the edges as shown (the bottom and right-hand ones are fixed to be $+1$ from the boundary condition on Z). Then τ_1 denotes the n spins $\sigma_1,..., \sigma_n$, and τ_2 the n spins $\sigma_1',..., \sigma_n'$. Noting that $\sigma_1' = \sigma_1$ (both being the center spin), it follows that

$$A(\tau_1|\tau_2) \equiv A(\sigma_1,..., \sigma_n|\sigma_1',..., \sigma_n') = \delta_{\sigma_1,\sigma_1'} \sum \exp(-\beta H_{ur}) \tag{5}$$

where H_{ur} is the interaction Hamiltonian for the spins shown in Fig. 3 (including the boundaries), and the summation is over all spins associated with internal sites (denoted by filled circles in Fig. 3).

Clearly $A(\tau_1|\tau_2)$ can be regarded as the element (τ_1, τ_2) of a 2^n by 2^n matrix A. Similarly for B, C, and D. Then Eq. (4) can be written

$$Z = \operatorname{Tr}(ABCD) \tag{6}$$

These matrices can be thought of as describing the effect of adding a quarter, or corner, of the lattice. We call them *corner transfer matrices*. From (1) and (5) it is fairly easy to see that

$$A' = C = C' = A, \qquad B' = D = D' = B \tag{7}$$

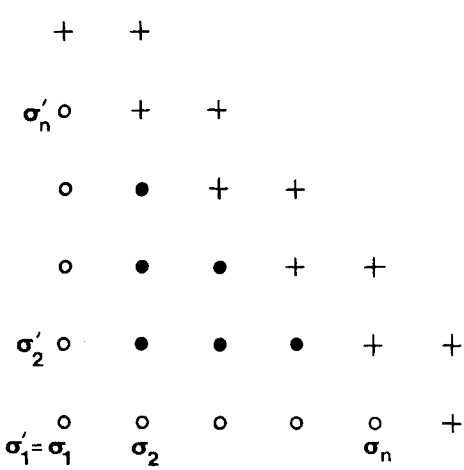

Fig. 3. The upper right corner of the lattice, corresponding to the CTM A. The summation in Eq. (5) is over those spins on sites shown by filled circles.

where A' is the transpose of A, etc. Also, B can be obtained from A by interchanging J and J'. Equations (6) can now be written

$$Z = \mathrm{Tr}(AB)^2 \tag{8a}$$

the free energy per site f is given by

$$-\beta f = \lim_{n\to\infty} [2n(n+1)]^{-1} \ln Z \tag{8b}$$

and the "partition function per site" κ by

$$\kappa = e^{-\beta f} = \lim_{n\to\infty} Z^{1/2n(n+1)} \tag{8c}$$

The spontaneous magnetization M can also be written in terms of A and B. As in (5), take σ_1 to be the center spin of the lattice. Then

$$M = \langle\sigma_1\rangle = \mathrm{Tr}[S(AB)^2]/\mathrm{Tr}(AB)^2 \tag{9}$$

where S is a diagonal matrix with elements

$$S(\sigma_1,\ldots,\sigma_n|\sigma_1',\ldots,\sigma_n') = \sigma_1\,\delta_{\sigma_1,\sigma_1'}\,\delta_{\sigma_2,\sigma_2'}\cdots\delta_{\sigma_n,\sigma_n'} \tag{10}$$

From (5), the matrix S commutes with both A and B.

3. HALF-ROW MATRICES

The corner transfer matrix (CTM) A is a product of matrices which successively build up the quadrant shown in Fig. 3 one row at a time. Consider the rows j and $j+1$ of spins shown in that figure, and label them as in Fig. 4. Then the contribution to the partition function of the interactions between these spins is

$$G_j(\sigma_1,\ldots,\sigma_n|\sigma_1',\ldots,\sigma_n') = \delta_{\sigma_1,\sigma_1'}\cdots\delta_{\sigma_j,\sigma_j'}\exp\Big[\beta\sum_{k=j}^{n}(J\sigma_{k+1}\sigma_{k+1}' + J'\sigma_k\sigma_{k+2}' + J_4\sigma_k\sigma_{k+1}\sigma_{k+1}'\sigma_{k+2}')\Big] \tag{11}$$

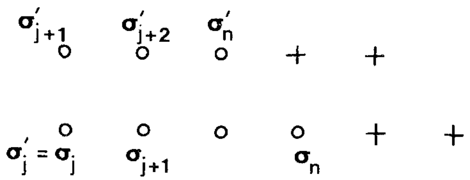

Fig. 4. The half-rows j and $j+1$ of Fig. 3, corresponding to the half-row matrix G_j. The right-hand side of Eq. (11) is the Boltzmann weight of the interactions between the spins on these two rows.

where $\sigma_{n+1} = \sigma'_{n+1} = \sigma'_{n+2} = +1$. The $G_j(\sigma_1,..., \sigma_n|\sigma_1',..., \sigma_n')$ can be regarded as elements of a 2^n by 2^n matrix G_j, and from Figs. 3 and 4 it can be seen that

$$A = G_1 G_2 G_3 \cdots G_n \tag{12}$$

Further, each G_j is a product of matrices that correspond to adding a square to the half-row shown in Fig. 4. From (11),

$$G_j = V_n V_{n-1} V_{n-2} \cdots V_j \tag{13}$$

where, for $j = 1,..., n$,

$$\begin{aligned} V_j(\sigma_1,..., \sigma_n|\sigma_1',..., \sigma_n') = {} & \delta_{\sigma_1,\sigma_1'} \cdots \delta_{\sigma_j,\sigma_j'} \delta_{\sigma_{j+2},\sigma'_{j+2}} \cdots \delta_{\sigma_n,\sigma_n'} \\ & \times \exp[\beta(J\sigma_{j+1}\sigma'_{j+1} + J'\sigma_j\sigma'_{j+2} \\ & + J_4\sigma_j\sigma_{j+1}\sigma'_{j+1}\sigma'_{j+2})] \end{aligned} \tag{14}$$

taking

$$\sigma_{n+1} = \sigma'_{n+1} = \sigma_{n+2} = \sigma'_{n+2} = +1 \tag{15}$$

The V_j can be written more neatly in operator form. Let s_j, c_j, and d_j be the Pauli operators

$$s_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad c_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad d_j = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{16}$$

acting on the jth spin. Also, let a, b, c, and d be the weights of the eight-vertex model, defined by[1,7]

$$\begin{aligned} a &= \exp[\beta(J + J' + J_4)], \qquad & b &= \exp[\beta(-J - J' + J_4)] \\ c &= \exp[\beta(-J + J' - J_4)], \qquad & d &= \exp[\beta(J - J' - J_4)] \end{aligned} \tag{17}$$

Then

$$V_j = \tfrac{1}{2}[a + d + (a - d)s_j s_{j+2} + (b + c)c_{j+1} + (c - b)s_j c_{j+1} s_{j+2}], \qquad j = 1,..., n - 1 \tag{18a}$$

$$V_n = \tfrac{1}{2}[a + d + (a - d)s_n]$$

(taking $s_{n+1} = +1$). Further, from (10) and (16),

$$S = s_1 \tag{19a}$$

We can now regard the CTM A as defined by (12), (13), (17), and (18a), and B as defined by a similar set of equations with J and J' (and hence c and d) interchanged. Noting from (18a) that each V_j is symmetric, and

$$V_j V_k = V_k V_j \qquad \text{if} \quad |j - k| \geqslant 2 \tag{20}$$

it is not difficult to verify from (12) and (13) that A and B are also symmetric operators.

Hereafter we shall regard a, b, c, and d as independent parameters, which is equivalent to introducing an arbitrary normalization factor (the same for A and B) into (17).

4. ALTERNATIVE REPRESENTATIONS

Since we have until now worked with the spin version of the eight-vertex model, we shall call the above representation (18a) and (19a) of the V_j and S the *spin representation*. There are two other representations that we shall use. They correspond to merely rearranging the rows and columns of the matrices.

4.1. Arrow Representation

Replace the indices $\sigma_1,..., \sigma_n$, $\sigma_1',..., \sigma_n'$ in (10) and (14) by $\mu_1,..., \mu_n$, $\mu_1',..., \mu_n'$, where

$$\begin{aligned} \mu_j &= \sigma_j\sigma_{j+1}, \qquad \mu_j' = \sigma_j'\sigma_{j+1}', \qquad j = 1,..., n-1 \\ \mu_n &= \sigma_n, \qquad \mu_n' = \sigma_n' \end{aligned} \tag{21}$$

Thus $\mu_j = +1$ if the corresponding adjacent spins are equal, -1 if they are different. The μ_j therefore describe the arrow states of the arrow formulation of the eight-vertex model.[1,7]

Now associate the Pauli operators (16) with μ_j, μ_j'. Then from (14) and (10),

$$V_j = \tfrac{1}{2}[a + d + (a - d)s_js_{j+1} + (b + c)c_jc_{j+1} + (b - c)d_jd_{j+1}], \qquad j = 1,..., n-1 \tag{18b}$$

$$V_n = \tfrac{1}{2}[a + d + (a - d)s_n]$$

$$S = s_1s_2s_3 \cdots s_n \tag{19b}$$

This representation has the merit of making the commutation relations (20) obvious. Also, since there are similarity transformations that permute (s_j, c_j, d_j) for all j, it clarifies the various symmetries of the eight-vertex model[8]: They correspond to interchanging the coefficients in the first of Eqs. (18b).

4.2. Third Representation

Replace $\sigma_1,..., \sigma_n, \sigma_1',..., \sigma_n'$ by $\nu_1,..., \nu_n, \nu_1',..., \nu_n'$, where

$$\nu_j = \sigma_j\sigma_{j+2}, \qquad \nu_j' = \sigma_j'\sigma_{j+2}', \qquad j = 1,..., n-2 \tag{22}$$

$$\nu_{n-1} = \sigma_{n-1}, \qquad \nu_{n-1}' = \sigma_{n-1}', \qquad \nu_n = \sigma_n, \qquad \nu_n' = \sigma_n' \tag{23}$$

Writing (14) and (10) in terms of the ν_j, ν_j', then converting to operator notation, we obtain

$$\begin{aligned} V_1 &= \tfrac{1}{2}[a + d + (a - d)s_1 + (b + c)c_2 + (c - b)s_1c_2] \\ V_j &= \tfrac{1}{2}[a + d + (a - d)s_j + (b + c)c_{j-1}c_{j+1} + (c - b)c_{j-1}s_jc_{j+1}], \\ &\qquad j = 2,..., n - 1 \quad (18c) \\ V_n &= \tfrac{1}{2}[a + d + (a - d)s_n] \end{aligned}$$

$$S = s_1s_3s_5 \cdots s_{n'} \tag{19c}$$

where $n' = n - 1$ or n according to whether n is even or odd, respectively.

This representation is useful when developing expansions in powers of b and c, since to leading order each V_j is a diagonal operator acting only on the jth "ν-spin."

We could draw up a table giving the various individual operators (e.g., c_j) in the three representations, but all the required information can be deduced by comparing (18a), (18b), and (18c).

5. "LOW-TEMPERATURE" EXPANSION

At low temperatures the dominant vertex weight in (17) is a, the others being much smaller. It is therefore appropriate to expand (18), (13), and (12) in powers of b, c, and d. However, from (18), V_j is diagonal when b and c are zero, so it is possible to expand in powers of b and c only, the coefficients being functions of a and d.

We shall be interested in obtaining the eigenvalues of A. It turns out that a useful first step is to evaluate the operator $\ln A$.

Use the third representation. Define

$$\alpha = \ln a, \qquad \delta = \ln d \tag{24}$$

and let h_j and u_j be the diagonal operators

$$h_j = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}, \qquad u_j = \begin{pmatrix} c/a & 0 \\ 0 & b/d \end{pmatrix} \tag{25}$$

acting on the jth "ν-spin." Let

$$\epsilon_1 = u_1c_2; \qquad \epsilon_j = c_{j-1}u_jc_{j+1}, \quad j = 2,..., n - 1; \qquad \epsilon_n = 0 \tag{26}$$

Then (18c) can be written

$$V_j = e^{h_j}(1 + \epsilon_j), \qquad j = 1,..., n \tag{27}$$

Substitute this into (13) and (12), expand to second order in the ϵ_j, and take the logarithm to this order. Define

$$\theta = (\alpha - \delta)/\sinh(\alpha - \delta), \qquad \tau = \{\sinh[2(\alpha - \delta)] - 2\alpha + 2\delta\}/4\sinh^2(\alpha - \delta) \tag{28}$$

After a fair amount of work one obtains

$$\ln A = \sum_{j=1}^{n} jh_j + \theta \sum_{j=1}^{n-1} j\epsilon_j + \tfrac{1}{2} \sum_{j=1}^{n-1} \epsilon_j^2[-j - (j-1)j\tau s_{j-1} + j(j+1)\tau s_{j+1}] + O(\epsilon^3) \tag{29}$$

A large number of unexpected simplifications occur in obtaining (29): The coefficients threaten to be quite complicated functions of j, but after many cancellations become only simple linear and quadratic forms; one expects terms containing $\epsilon_j\epsilon_{j+1}$ and $\epsilon_j\epsilon_{j+2}$, but these cancel out.

Now we look for an orthogonal operator P that transforms $\ln A$ and A to diagonal form, i.e.,

$$P^T A P = A_d, \qquad P^T P = 1 \tag{30}$$

where A_d is diagonal.

Note that to leading order in the (b, c) expansion,

$$\ln A = \sum_{j=1}^{n} jh_j \tag{31}$$

Many of the eigenvalues of this operator are degenerate, so one might expect a higher order calculation to resolve this degeneracy, and that even to leading order we could not make the obvious choice $P = 1$. However, to second order this turns out not to be the case, and degenerate eigenvalues remain degenerate.

Define two sets of real antisymmetric operators p_j and q_j by

$$\begin{aligned} p_1 &= -\tfrac{1}{2} iad(a^2 - d^2)^{-1} u_1 d_2 \\ q_1 &= 0 \\ p_j &= -\tfrac{1}{2} iad(a^2 - d^2)^{-1} c_{j-1} u_j d_{j+1} \\ q_j &= -\tfrac{1}{2} iad(a^2 - d^2)^{-1} d_{j-1} u_j c_{j+1} \end{aligned} \tag{32}$$

where $j = 2,..., n-1$. Writing $(x, y) = xy - yx$ for the commutator of two operators, to second order we find that

$$\begin{aligned} \ln P = {} & \sum_{j=1}^{n-1} [(j+1)p_j - (j-1)q_j] \\ & - \sum_{j=1}^{n-1} [(j+2)(p_j, p_{j+1}) + (2j+1)(p_j, q_{j+1}) + (j-1)(q_j, q_{j+1})] \\ & - \tfrac{1}{2} \sum_{j=1}^{n-3} [(j+3)(p_j, p_{j+2}) + (j-1)(q_j, q_{j+2})] + O(\epsilon^3) \end{aligned} \tag{33}$$

$$\ln A_d = \sum_{j=1}^{n} jh_j + \tfrac{1}{2} \sum_{j=1}^{n-1} \epsilon_j^2[-j - (j-1)j\tau' s_{j-1} + j(j+1)\tau' s_{j+1}] + O(\epsilon^3) \tag{34}$$

where

$$\tau' = \tfrac{1}{2}(a^2 + d^2)/(a^2 - d^2) \tag{35}$$

The expressions (29) and (34) can be further simplified. Define m_j to be the operator

$$m_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{36}$$

acting on spin j. Then from (16), (25), and (26),

$$s_j = 1 - 2m_j \tag{37}$$

$$\epsilon_j{}^2 = u_j{}^2 = \frac{c^2}{a^2} + \left(\frac{b^2}{d^2} - \frac{c^2}{a^2}\right)m_j \tag{38}$$

Substituting these expressions into the last summations in (29) and (34), we find that many of the terms in the summation cancel, leaving

$$\ln A = -Cn + \sum_{j=1}^{n} jg_j + \theta \sum_{j=1}^{n-1} j\epsilon_j - R_n + O(\epsilon^3) \tag{39}$$

$$\ln A_d = -C'n + \sum_{j=1}^{n} jg_j' - R_n' + O(\epsilon^3) \tag{40}$$

where

$$C = (\tau - \tfrac{1}{2})c^2/a^2 \tag{41}$$

$$g_j = h_j + (\tau - \tfrac{1}{2})u_j{}^2 + 2c^2\tau m_j/a^2 \tag{42}$$

$$\begin{aligned} R_n = {} & \frac{c^2}{a^2}\tau[(n-1)nm_{n-1} + n(n+1)m_n] \\ & + \left(\frac{b^2}{d^2} - \frac{c^2}{a^2}\right)[(\tau - \tfrac{1}{2})nm_n + \tau(n-1)nm_{n-1}m_n] \end{aligned} \tag{43}$$

and C', g_j', and R_n' are defined similarly, but with τ replaced by τ'. The g_j, g_j', R_n, and R_n' are diagonal operators; g_j and g_j' act only on spin j.

Note that R_n and R_n' are "remainder" terms arising from the last term(s) in the summations over j.

In this calculation we have retained all terms of order b, c, b^2, bc, and c^2 and neglected cubic and higher terms. We have made no assumption regarding the relative magnitudes of a and d.

Suppose now that J and J' are both large and positive but $J - J'$ and J_4

are of order unity or less. Then from (17) there exists a small parameter t such that

$$c/a \sim t, \qquad d/a \sim t, \qquad b/a \sim t^2 \tag{44}$$

In this case, each operator V_j is singular (i.e., has zero determinant) to leading order. The expansion of $\ln A$ is therefore not trivial. Nevertheless, note that, from (25) and (26), u_j and ϵ_j are of order t. It seems that the other coefficients (e.g., θ and τ) in the (b, c) expansion are of order unity or less. Also, by performing direct calculations for small values of n, it appears that there exists a perturbation expansion of $\ln P$ in increasing powers of t^2 and expansions of $\ln A$ and $\ln A_d$ of the form

$$\sum_{j=1}^{n} jh_j + (\delta - \alpha)(t^2L_1 + t^4L_2 + \cdots) + t^2M_1 + t^4M_2 + \cdots \tag{45}$$

It follows that (33), (39), and (40) provide expansions valid to order t^2. Define

$$u_j' = a^{-1}du_j = a^{-2}\begin{pmatrix} cd & 0 \\ 0 & ab \end{pmatrix} \sim t^2 \tag{46}$$

and take $c_0 = 1$; then the expansions are

$$\ln P = -\tfrac{1}{2}i \sum_{j=1}^{n-1} [(j+1)c_{j-1}u_j'd_{j+1} - (j-1)d_{j-1}u_j'c_{j+1}] + O(t^4) \tag{47}$$

$$\ln A_d = \sum_{j=1}^{n} j(h_j + c^2m_j/a^2) - R_n + O(t^4) \tag{48}$$

$$\ln A = \ln A_d + 2(\alpha - \delta) \sum_{j=1}^{n-1} c_{j-1}u_j'c_{j+1} + O(t^4) \tag{49}$$

where R_n is still defined by (43), but with τ given its leading-order value of $\frac{1}{2}$.

5.1. Free Energy and Magnetization

The free energy and magnetization are given by (8) and (9). To use these expressions we must consider the other CTM B, which is obtained from A by interchanging c and d. From (44), if t is small for A, then it is also small for B, so we can consider an expansion of A, B, f, and M in increasing powers of t.

Further, from (46) and (47), to order t^2 the orthogonal matrix P that diagonalizes A is *unaltered by interchanging c and d.* Thus, at least to this order, A, B, and S can be simultaneously diagonalized, and in (8) and (9) we can replace A and B by their diagonal representations A_d and B_d. The products and traces are then trivial to evaluate.

From (48), (36), (25), and (24), to relative order t^2,

$$A_d = \rho_n \kappa^{n(n+1)/2}[\exp(-R_n)]\left[\begin{pmatrix}1 & 0\\ 0 & w\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & w^2\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & w^3\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1 & 0\\ 0 & w^n\end{pmatrix}\right] \tag{50a}$$

$$B_d = \rho_n' \kappa^{n(n+1)/2}[\exp(-R_n')]\left[\begin{pmatrix}1 & 0\\ 0 & w'\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & w'^2\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & w'^3\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1 & 0\\ 0 & w'^n\end{pmatrix}\right] \tag{50b}$$

where ρ_n, ρ_n', and κ are scalar factors given to this order by

$$\rho_n = \rho_n' = 1, \qquad \kappa = a \tag{51}$$

R_n is a diagonal operator given by (43) with $\tau = \frac{1}{2}$, R_n' is now R_n with c and d interchanged, and

$$w = a^{-1}d[1 + c^2/a^2 + O(t^4)], \qquad w' = a^{-1}c[1 + d^2/a^2 + O(t^4)] \tag{52}$$

In particular, for t small,

$$0 < w < 1, \qquad 0 < w' < 1 \tag{53}$$

$$\lim_{n\to\infty} n^{-2} \ln \rho_n = \lim_{n\to\infty} n^{-2} \ln \rho_n' = 0 \tag{54}$$

Now R_n and R_n' are diagonal operators acting on spins $n - 1$ and n only. Their elements are positive and (for large n) proportional to n^2, except when spins $n - 1$ and n are both up (equal to $+1$), when they are zero.

Substituting (50) into (8) and (9), it follows that *the contribution of R_n and R_n' to Z and M is negligible in the limit of n large.* Hence they can be ignored in the thermodynamic limit, leaving A_d and B_d as simple direct products of diagonal 2 by 2 matrices.

Applying the orthogonal transformation P leaves the center-spin operator S, given in this representation by (19c), unchanged. Substituting the expressions (50) and (19c) into (8) and (9), neglecting R_n and R_n', and using *only* the conditions (53) and (54), we obtain in the limit of n large

$$\text{the } \kappa \text{ in (8c)} = \text{the } \kappa \text{ in (50)} \tag{55}$$

$$M = \prod_{j=1}^{\infty} [1 - (ww')^{4j-2}]/[1 + (ww')^{4j-2}] \tag{56}$$

Now using (51) and (52), we can verify that these expressions are indeed correct to the appropriate orders in the t expansion (t^2 and t^6, respectively).

Further, and most suggestively, the expression (56) is precisely of the form conjectured by Barber and Baxter[2] (on the basis of longer series expansions and the known Ising result), namely

$$M = \prod_{j=1}^{\infty} \frac{1 - x^{4j-2}}{1 + x^{4j-2}} \tag{57}$$

where x is the variable occurring in the elliptic function parametrization of the weights of the eight-vertex model (Section 6).

5.2. ln A as a Heisenberg-Type Operator

We remarked above that ln A (to second order in the expansions) was much simpler than might have been expected. This is more clearly seen in the arrow representation of Section 4, rather than the third representation used above. Using now the arrow representation, we find from (39), (42), (24)–(26), and (28) that to second order in the (b, c) expansion

$$\ln A = \phi_n + \tfrac{1}{2} \sum_{j=1}^{n-1} \mathscr{J}_1 j(s_j s_{j+1} + \mathscr{J}_2 c_j c_{j+1} + \mathscr{J}_3 d_j d_{j+1} + \mathscr{J}_4) - R_n'' \tag{58}$$

where

$$\begin{aligned} \mathscr{J}_1 &= \alpha - \delta - (\tau - \tfrac{1}{2})b^2/d^2 - (\tau + \tfrac{1}{2})c^2/a^2 \\ \mathscr{J}_2 &= 2(\alpha - \delta)(ab + cd)/(a^2 - d^2) \qquad\qquad (59) \\ \mathscr{J}_3 &= 2(\alpha - \delta)(ab - cd)/(a^2 - d^2) \end{aligned}$$

$$\mathscr{J}_4 = \alpha + \delta + (3\tau - \tfrac{1}{2})c^2/a^2 + (\tau - \tfrac{1}{2})b^2/d^2 \tag{60}$$

$$\phi_n = n\alpha \tag{61}$$

$$\begin{aligned} R_n'' = {} & (\alpha - \delta)nm_n + (c^2/a^2)\tau(n-1)n(m_{n-1} + m_n - m_{n-1}m_n) \\ & + (b^2/d^2)\tau(n-1)nm_{n-1}m_n \end{aligned} \tag{62}$$

In particular,

$$\lim_{n\to\infty} n^{-2}\phi_n = 0 \tag{63}$$

and to the obtained second order in the (b, c) expansion of $\mathscr{J}_1$, $\mathscr{J}_2$, $\mathscr{J}_3$, it is true that

$$\mathscr{J}_1 : \mathscr{J}_2 : \mathscr{J}_3 = \tfrac{1}{2}(a^2 + b^2 - c^2 - d^2) : (ab + cd) : (ab - cd) \tag{64}$$

The "remainder operator" R_n'' is diagonal, acts only on spins $n - 1$ and n, and has positive elements proportional to n^2 for n large, except for the element corresponding to both spins $n - 1$ and n being up, which element is zero. Thus R_n'' is negligible in the same sense as R_n and R_n' are in (50): Its contribution to the traces in (8) and (9) is negligible in the thermodynamic limit. Also, the scalar additive ϕ_n makes no contribution to the free energy or magnetization in this limit, since it satisfies (63).

Hence ϕ_n and R_n'' can be ignored when n is large, and we see from (58) that $\ln A$ is an operator of Heisenberg type[9] (remember that s_j, c_j, and d_j are the Pauli operators acting on spin j), but with coefficients proportional to j. It is known that the row-to-row transfer matrix of the eight-vertex model is intimately connected with a Heisenberg chain operator[10] whose coefficients satisfy (64), so (58) is suggestive of a similar connection for the corner transfer matrix.

6. ELLIPTIC FUNCTION PARAMETRIZATION

From the derivation of the free energy of the eight-vertex model[1] it is known that the system is in an ordered ferromagnetic state if

$$a > b + c + d \tag{65}$$

It undergoes a phase transition when $a = b + c + d$. (This is also suggested simply by the previously obtained symmetry relations.[11]) The "low-temperature" expansions discussed above lie in the regime (65), provided $a > d$.

It is also known that an elliptic function parametrization of a, b, c, and d occurs naturally in the derivation of the free energy. In the regime (65) this parametrization is to define k, η, and v such that

$$a:b:c:d = \operatorname{sn} 2\eta : [-k \operatorname{sn} 2\eta \operatorname{sn}(\eta - v) \operatorname{sn}(\eta + v)] : \operatorname{sn}(\eta + v) : \operatorname{sn}(\eta - v) \tag{66}$$

where $\operatorname{sn} u \equiv \operatorname{sn}(u, k)$ is the elliptic sn function[12] of argument u and modulus k.

In the regime (65), k is real, η and v are pure imaginary, and

$$0 < k < 1, \qquad |\operatorname{Im} v| < \operatorname{Im} \eta < \tfrac{1}{2}K' \tag{67}$$

where K and K' are the complete elliptic integrals of the first kind, of moduli k and $k' = (1 - k^2)^{1/2}$, respectively.

The function $\operatorname{sn} u$ can be written as an infinite product of elementary functions. Let q be the nome of the elliptic functions, given by

$$q = \exp(-\pi K'/K) \tag{68}$$

Then, from §8, 146.23 of Ref. 12 (with $\sqrt{q}$ corrected to $q^{1/4}$), $\operatorname{sn} u$ is given by

$$k^{1/2} \operatorname{sn} u = 2q^{1/4} \sin \frac{\pi u}{2K} \prod_{j=1}^{\infty} \frac{(1 - q^{2j} e^{i\pi u/K})(1 - q^{2j} e^{-i\pi u/K})}{(1 - q^{2j-1} e^{i\pi u/K})(1 - q^{2j-1} e^{-i\pi u/K})} \tag{69}$$

From (66) and (69), the ratios $a:b:c:d$ can be written as infinite products of polynomial functions of q and

$$x = e^{i\pi\eta/K}, \qquad z = e^{i\pi v/K} \tag{70}$$

Since η and v are pure imaginary, x and z are real and positive, and

$$x < z < x^{-1}, \qquad 0 < q < x^2 < 1 \tag{70a}$$

By using various elliptic function identities, one can establish that

$$(a^2 + b^2 - c^2 - d^2)/cd = 2 \operatorname{cn} 2\eta \operatorname{dn} 2\eta, \qquad ab/cd = -k \operatorname{sn}^2(2\eta) \tag{71}$$

Note that these quantities, and therefore the ratios in (64), depend on k and η, but are independent of v.

We can write the results of our expansions in terms of q, x, and z, instead of a, b, c, and d. In the (b, c) expansion we took

$$a \sim d \gg b \sim c \tag{72}$$

from which it follows that

$$x \ll 1, \qquad q \sim x^4, \qquad z \sim x^{-1}, \qquad b/a \sim c/a \sim x \tag{73}$$

Alternatively, when t in (44) is small, we find that

$$x \ll 1, \qquad q \sim x^4, \qquad z \sim 1, \qquad t \sim x^{1/2} \tag{74}$$

In either case, substituting the expression (69) into (66), using (70), and neglecting terms of relative order smaller than x^2, we obtain

$$\begin{aligned} b/a &= x^{-1}q^{1/2}(1 - xz)(1 - xz^{-1})/[(1 - qzx^{-1})(1 - qz^{-1}x^{-1})] \\ c/a &= (x/z)^{1/2}(1 - xz)(1 - x^{-2}q)/[(1 - x^2)(1 - qz^{-1}x^{-1})] \\ d/a &= (xz)^{1/2}(1 - xz^{-1})(1 - x^{-2}q)/[(1 - x^2)(1 - qzx^{-1})] \end{aligned} \tag{75}$$

Substituting these expressions into (59) and (60), to second order in the (b, c) expansion we find

$$\mathscr{I}_1 = -\tfrac{1}{2}[\ln(xz)][1 + 2x^2 + 2qx^{-2} + O(x^3)] \tag{76}$$

$$\mathscr{I}_4 = -\mathscr{I}_1 + 2[\alpha + x^2(1 - xz)] + 2x^2 \ln(xz) + O(x^3) \tag{77}$$

while on substituting them into (52), to second order in the t expansion we obtain

$$\begin{aligned} w &= (xz)^{1/2}[1 + 0\cdot x + O(x^2)] \\ w' &= (x/z)^{1/2}[1 + 0\cdot x + O(x^2)] \end{aligned} \tag{78}$$

In particular, note from (78) that, to the order obtained, $ww' = x$. Hence the two expressions (56) and (57) for the magnetization certainly agree to this order.

7. CONJECTURES

We have considered the "low-temperature" expansions of the matrices in some detail, because these calculations can be performed explicitly for finite n. They indicate that $\ln A$ and $\ln A_d$ may have quite simple forms, except for:

(a) An additive scalar ϕ_n that satisfies (63), and hence gives zero contribution to the free energy [Eq. (8)] in the limit of n large, and cancels out of Eq. (9).

(b) A remainder operator $-R_n$, which gives zero contribution to either (8) and (9) in the limit of n large. To some order r in perturbation theory, this may be expected to act only on spins $n - r + 1$, $n - r + 2, \ldots, n$.

We now make the following conjectures, all of which apply only in the ferromagnetic regime (65), and are consistent with the above expansions.

(i) The orthogonal matrices that diagonalize the two CTM's A and B are the same, apart from a factor of type $\exp(-R_n)$: i.e., a factor that can be ignored (in some appropriate sense) in the limit of n large. The consequence of this is that A and B can be diagonalized simultaneously, together with S. The products and traces in (8) and (9) then become simple to evaluate.

(ii) In the arrow representation, the operator $\ln A$ is of the form (58), where ϕ_n and R_n' are negligible in the sense described above [in particular, ϕ_n satisfies (63)], and

$$\begin{aligned} \mathscr{J}_1 &= \tfrac{1}{2}\xi(a^2 + b^2 - c^2 - d^2)/(abcd)^{1/2} \\ \mathscr{J}_2 &= \xi(ab + cd)/(abcd)^{1/2} \\ \mathscr{J}_3 &= \xi(ab - cd)/(abcd)^{1/2} \end{aligned} \tag{79}$$

where ξ is real and positive, and is defined in terms of the elliptic parametrization of Section 6 by

$$\begin{aligned} \xi = -ik^{1/2}(v + \eta) &= \pi^{-2}k^{1/}\tfrac{1}{2}K\ln[(xz)^{-1}] \\ &= q^{1/4}\ln[(xz)^{-1}]\prod_{j=1}^{\infty}\left(\frac{1 - q^{2j}}{1 - q^{2j-1}}\right)^2 \end{aligned} \tag{80}$$

Corollaries: Since B is obtained from A by interchanging the Boltzmann weights c and d, a similar conjecture applies for $\ln B$, but with v negated, and z inverted, in (80).

Apart from the negligible R_n'' contribution, $\ln A$ is of the form

$$\ln A = \text{scalar} - \xi\mathscr{H} \tag{81}$$

where the operator $\mathscr{H}$ depends on a, b, c, and d only via the ratios (71). The eigenvector matrix P therefore depends only on these ratios. In particular, it is unaltered by interchanging c and d, so is the same for both A and B. Thus conjecture (i) follows from (ii).

(iii) Let $\mathscr{H}_{\text{lin}}$ be the anisotropic Heisenberg chain operator with constant coefficients:

$$\mathscr{H}_{\text{lin}} = -\tfrac{1}{2} \sum_{j=1}^{n-1} (\mathscr{J}_1 s_j s_{j+1} + \mathscr{J}_2 c_j c_{j+1} + \mathscr{J}_3 d_j d_{j+1}) \tag{82}$$

where $\mathscr{J}_1$, $\mathscr{J}_2$, and $\mathscr{J}_3$ are given by (79). Let $\Lambda_{\min}$ be its minimum eigenvalue, and let

$$F = \lim_{n \to \infty} \Lambda_{\min}/2n \tag{83}$$

Then we conjecture that the fourth coefficient in (58) is

$$\mathscr{J}_4 = 2 \ln \kappa + 4F \tag{84}$$

where κ is given by (8c).

The quantities κ and F are known from the original results for the eight-vertex model.[1,10] In the regime (65) which we are considering, where

$$\mathscr{J}_1 > \mathscr{J}_2 > |\mathscr{J}_3| \tag{85}$$

they are given by

$$\ln \frac{\kappa}{a} = \sum_{j=1}^{\infty} \frac{x^{-j}(x^{2j} - q^j)^2 (x^j + x^{-j} - z^j - z^{-j})}{j(1 - q^{2j})(1 + x^{2j})} \tag{86}$$

$$F = -\frac{1}{4} \mathscr{J}_1 - \frac{1}{2} i k^{-1/2} \xi \left(\frac{d}{dv} \ln \frac{\kappa}{a} \right)_{v = -\eta} \tag{87}$$

$$= -\frac{1}{4} \mathscr{J}_1 + \frac{1}{2} \ln(xz) \sum_{j=1}^{\infty} \frac{x^{-2j}(x^{2j} - q^j) \frac{1}{2} (1 - x^{2j})}{(1 - q^{2j})(1 + x^{2j})} \tag{88}$$

the differentiation in (87) being performed for fixed k, η, q, and x.

(iv) We conjecture that (in the third representation of Section 4) the CTM's A and B have the simple diagonal form given in (50), i.e., a direct product of two by two matrices, together with factors ρ_n and $\exp(-R_n)$ which are negligible in the thermodynamic limit, that (53) and (54) are satisfied, and that

$$w = (xz)^{1/2}, \qquad w' = (x/z)^{1/2} \tag{89}$$

Corollaries: Substituting (50) into (8), we obtain Eq. (55). Thus κ in (50) must be the partition function per site.

Substituting (50) into (9), we obtain expression (57) for the spontaneous magnetization, previously conjectured by Barber and Baxter.[2]

Taking logarithms in (50a), applying the inverse orthogonal transformation P^{-1} to obtain $\ln A$, and assuming [as is implied by conjecture (ii)] that P depends on q and x, but not z, we obtain (ignoring terms negligible for n large)

$$\ln A = \tfrac{1}{2}n(n+1)\ln \kappa + [\ln(xz)]W \tag{90}$$

where the operator W is independent of z. Taking the limit $z \to x^{-1}$ (i.e., $v \to -\eta$), b and c become small, and we can use our perturbation expansion result (58)–(60) to evaluate W. The result obtained is that given in conjectures (ii) and (iii).

Thus conjecture (iv), together with the independence of P on z, implies the three previous conjectures.

8. CONCLUSIONS

If the above conjectures are correct, then the corner transfer matrices may well provide a useful tool for handling exactly soluble models in the thermodynamic limit, since then they have an extremely simple diagonal form.

To do this properly it would be necessary to set up appropriate Banach algebras to handle the resulting infinite-dimensional operators. Presumably these algebras would be related to the C^* algebra.[13] Here we have avoided this problem.

In all previous work q, x, and z have been rather buried in the mathematics, though it has been remarked that they are "natural" parameters to use.[4] However, now we can give x and z a direct physical definition, for from (50) and (89)

$$x = \Lambda_{A,1}\Lambda_{B,1}/\Lambda_{A,0}\Lambda_{B,0} \tag{91}$$

$$z = \Lambda_{A,1}\Lambda_{B,0}/\Lambda_{A,0}\Lambda_{B,1} \tag{92}$$

where $\Lambda_{A,0}$ and $\Lambda_{A,1}$ ($\Lambda_{B,0}$ and $\Lambda_{B,1}$) are the largest and next largest, respectively, eigenvalues of the corner transfer matrix A (B).

All the conjectures of course agree with the second-order perturbation expansions obtained. This evidence may not seem very convincing to the reader, but if the calculations are worked through in detail the number of cancellations needed, and that occur, for this to happen becomes quite impressive.

More significantly, we can in fact prove conjectures (i) and (ii) in general, and all four conjectures for the Ising model case, when $ab = cd$, subject to a nonrigorous treatment of the limit of n large. It is intended to publish these results shortly.

It is interesting to note that our conjecture (iv) implies that any operator of the form (58) has the diagonal form (in the arrow representation)

$$\frac{1}{2}\sum_{j=1}^{n-1} j(\lambda s_j s_{j+1} + \mu) \tag{93}$$

where λ and μ are scalars and we have ignored negligible contributions of type ϕ_n, R_n. Thus all such operators are related to one another, within additive and multiplicative scalars, by orthogonal similarity transformations. These transformations must form a group, and it may be that a way to prove conjecture (iv) would be to examine this group.

One of the delightful properties of the Ising model is that *all* the 2^n by 2^n matrices that occur form a group, and that the members of this group can be represented by $2n$ by $2n$ matrices.[14] Unfortunately, no such property is known for the general six- and eight-vertex models. A rather ambitious hope is that by examining the CTM's we may stumble on such a group, that the solution of the models may thereby be simplified, and even that it may then be possible to solve new models, such as a staggered eight-vertex model.

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